The Form of Even Perfect Numbers— an Independent Derivation

by

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Abstract

This article presents my derivation of the general form of even perfect numbers which is $[2^{n-1} \times (2^n - 1)]$, where $(2^n - 1)$ is a prime number— a derivation which I have pursued perfectly independently, being perfectly unaware of the ways of doing it of Euler or of any other Mathematician. Thus, I show here that every even perfect number must be of the form $[2^{n-1} \times (2^n - 1)]$, where $(2^n - 1)$ is a prime number. The proof here has an element of simplicity in it. The proof starts from the definition of an even number and that of a perfect number, and proceeds by small logical steps, gradually evolving in a perfectly straightforward manner. It first identifies that there are three forms into which even numbers may be classified. It then considers these classes one by one, and tests for each whether the even numbers belonging to that class qualify to ever be perfect or not. The classes get eliminated one by one, and we are finally left with only one class— that class the members of which have the form $\left[2^{(an\ integer\ )} \times (a\ prime\ number)\right]$. Now, putting the members of this class to the test of perfectness, we find that those alone qualify which have the prime factor expressible as $\left[2^{(the\ same\ integer\ +\ 1)} - 1\right]$. We thus reach the end of our course. The derivation, thus, stands out in its simplicity. Another point that is interesting about this derivation is that it, along its path, makes a passing glance with the problem regarding odd perfect numbers of whether they exist or not— at equality (7), but admittedly, does not make any progress in solving it.

1. OVERVIEW

The ancient Greek Mathematicians had proven that if $\left(2^n - 1\right)$ is a prime number, then $\left\{2^{n-1} \times \left(2^n - 1\right)\right\}$ is a perfect number. About two thousand years later, Euler proved the converse of this theorem, i.e., that every even perfect number must be of the above form.

Here, I present my own proof of the fact that every even perfect number must be of the form $\left\{2^{n-1} \times \left(2^n - 1\right)\right\}$, where $\left(2^n - 1\right)$ is a prime number. The proof has been pursued perfectly independently, i.e., being perfectly unaware of the ways of proving it of Euler or of any other Mathematician.

2. INTRODUCTION

Let me first state what perfect numbers are. A perfect number is such a number which is equal to the sum of its factors greater than or equal to 1 but less than itself. E.g.— 6 (6 = 1 + 2 + 3), 28 (28 = 1 + 2 + 4 + 7 + 14).
Note that

$$
\begin{aligned}
6 &= 1 + 2 + 3 \\
&= \left(1 + 2^1\right) + 3 \times 1 \\
&= \left(2^2 - 1\right) + \left(2^2 - 1\right) \times 1 \quad \left[\because 1 + 2^1 = 3 = 2^2 - 1\right] \\
&= 2 \times \left(2^2 - 1\right) \\
&= 2^{2-1}\left(2^2 - 1\right)
\end{aligned}
$$

$$
\begin{aligned}
28 &= 1 + 2 + 4 + 7 + 14 \\
&= \left(1 + 2^1 + 2^2\right) + 7\left(1 + 2^1\right) \\
&= \left(2^3 - 1\right) + \left(2^3 - 1\right)\left(2^2 - 1\right) \quad \left[\because 1 + 2^1 + 2^2 = 7 = 2^3 - 1\right] \\
&= \left(2^3 - 1\right)\left(1 + 2^2 - 1\right) \\
&= 2^2\left(2^3 - 1\right) \\
&= 2^{3-1}\left(2^3 - 1\right)
\end{aligned}
$$

Observations such as these had led the ancient Greek Mathematicians to prove that if $\left(2^n - 1\right)$ is a prime number, then $\left\{2^{n-1} \times \left(2^n - 1\right)\right\}$ is a perfect number. Observe that in

the above examples, $(2^2-1)=3$ and $(2^3-1)=7$, which are prime numbers. Later, Euler proved the converse of this theorem, i.e., that every even perfect number must be of the above form.

However, no one has ever discovered an odd perfect number. No one even knows whether an odd perfect number exists or not.

What I present here is my own proof of the theorem that every even perfect number must be of the form

$$2^{n-1} \times (2^n - 1),$$

where $(2^n - 1)$ is prime.

3. The Theorem To Be Proved

Every even perfect number must be of the form

$$2^{n-1} \times (2^n - 1),$$

where $(2^n - 1)$ is a prime number.

4. The Proof

An even number can have any one of the following forms:—

(a) 2^n, where n is a natural number

(b) $2^n \prod\limits_{i=1}^{m} p_i^{l_i}$, where m and n are natural numbers,

$\quad\quad\quad$ p_i is a prime number other than 2, $\forall i = 1(1)m$,

$\quad\quad\quad$ l_i is a natural number, $\forall i = 1(1)m$,

$\quad\quad\quad$ $p_i \neq p_j$ for $i \neq j$,

$\quad\quad\quad$ l_i & l_j ($i \neq j$) may be same or different.

No even number of the form (a) can be a perfect number. $[\because 1 + 2^1 + 2^2 + \cdots + 2^{n-1} = 2^n - 1 \neq 2^n]$

Now, let us consider the even numbers of the form (b) which satisfy the condition that if $m = 1$, then $l_1 \neq 1$.

Let, $\prod\limits_{i=1}^{m} p_i^{l_i} = P$ ………. (1) [if $m = 1$, then $l_1 \neq 1$]

If a number of the form (b) (which satisfies the condition that if $m = 1$, then $l_1 \neq 1$) has to be perfect, it must satisfy the equation

$2^n P = \left(1 + 2^1 + 2^2 + \cdots + 2^n\right) + \left(1 + 2^1 + 2^2 + \cdots + 2^n\right) \times$ (sum of all the factors
$\quad\quad$ of P that are greater than 1 but less than itself)
$\quad\quad + P\left(1 + 2^1 + 2^2 + \cdots + 2^{n-1}\right)$ ………. (2)

or, $2^n P = \left(1 + 2^1 + 2^2 + \cdots + 2^n\right) \times$ [1 + sum of all the factors of P that are greater than
$\quad\quad$ 1 but less than itself] $+ P\left(1 + 2^1 + 2^2 + \cdots + 2^{n-1}\right)$

$\quad\quad = \left(2^{n+1} - 1\right) \times$ (sum of all the factors of P that are greater than or equal to 1
$\quad\quad$ but less than itself) $+ P\left(2^n - 1\right)$

$\quad\quad = \left(2^{n+1} - 1\right)S + P\left(2^n - 1\right),$

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where S = sum of all the factors of P that are greater than or equal to 1 but less than itself ………. (3)

or, $2^n P = 2^n P + \left[\left(2^{n+1} - 1 \right) S - P \right]$ ……….. (4)

Equation (4) holds true if and only if

$$\left(2^{n+1} - 1 \right) S - P = 0$$

$$\text{or, } P = \left(2^{n+1} - 1 \right) S$$

$$\text{or, } \frac{P}{S} + 1 = 2^{n+1}$$

$$\text{or, } Q + 1 = 2^{n+1} \text{ ………… (5),}$$

where $Q = \dfrac{P}{S}$ ……….. (6)

We note that equation (5) cannot hold true if Q is not an integer. Let us investigate whether Q is an integer or not. From equation (6), it is evident that for Q to be an integer, S must be equal to any one of the factors of P. But from equation (3), it is evident that S cannot be equal to any one of the factors of P that are greater than or equal to 1 but less than P. Hence, there is only one possibility for which Q can be an integer and that is

$$S = P \text{ ………. (7)}$$

Let us note that if for some P, equation (7) holds true, then that P is an odd perfect number. However, we shall not go to investigate whether equation (7) is possible or not, since we do not need to do so for the current problem. If at all equation (7) holds true for some value of P, then using equations (5), (6) & (7), we have

$$1 + 1 = 2^{n+1}$$

$$\text{or, } 2^1 = 2^{n+1},$$

which is impossible for $n \geq 1$.

Thus, we see that whatever be the value of P, equation (5) does not hold true for $n \geq 1$.

Hence, an even integer of the form (b) (which satisfies the condition that if $m = 1$, then $l_1 \neq 1$), cannot be perfect.

Now, we are left with only one form of even numbers and that is

$$2^n p \text{ ………. (c),}$$

where n is a natural number

and p is a prime number other than 2.

Let us now investigate whether a number of the form (c) can be perfect or not.

If such a number has to be perfect, it must satisfy the equation

$$2^n p = \left(1 + 2^1 + 2^2 + \cdots + 2^n \right) + p \left(1 + 2^1 + 2^2 + \cdots + 2^{n-1} \right) \text{ ………… (8)}$$

$$\text{or, } 2^n p = \left(2^{n+1} - 1 \right) + p \left(2^n - 1 \right)$$

$$\text{or, } 2^n p = 2^n p + \left[\left(2^{n+1} - 1 \right) - p \right] \text{…………. (9)}$$

Equation (9) holds true if and only if

$$\left(2^{n+1} - 1 \right) - p = 0$$

$$\text{or, } p = 2^{n+1} - 1 \text{ ………….. (10)}$$

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We note that certain numbers of the form $\left(2^{n+1}-1\right)$ corresponding to certain values of n are prime. E.g.— $2^{1+1}-1=3$ ($n=1$) , $2^{2+1}-1=7$ ($n=2$). It is also obvious that a number of this form can never be equal to 2 for $n\geq 1$. Hence, equation (10) is perfectly legitimate. Thus, an even number of the form (c) is perfect if and only if the prime number p is of the form $\left(2^{n+1}-1\right)$.

We have also already shown that even numbers that are not of the form (c) cannot be perfect.

Hence, every even perfect number must be of the form

$$2^n\left(2^{n+1}-1\right),$$

where n is a natural number
   and $\left(2^{n+1}-1\right)$ is prime.

Written in another form, every even perfect number must be of the form

$$2^{n-1}\left(2^n-1\right),$$

where n is a natural number greater than 1
   and $\left(2^n-1\right)$ is prime.

This completes the proof.

5. CONCLUSION

We have thus derived that every even perfect number must be of the form

$$2^{n-1}\left(2^n-1\right),$$

where n is a natural number greater than 1
   and $\left(2^n-1\right)$ is prime.

We may take note of the simplicity that underlies this derivation. As we have seen, the derivation starts by classifying even numbers, then tests these classes for perfectness, and in doing so, eliminates them one by one only to be left with one class of even numbers, that class the members of which are of the form $\left[2^{(an\ integer\ )}\times p\right]$, where p is a prime number, and putting these members to the test of perfectness, it finds that those numbers alone qualify to be perfect which are of the form $\left\{2^{n-1}\times\left(2^n-1\right)\right\}$ where $\left(2^n-1\right)$ is a prime number. It thus finds the form that all even perfect numbers must be of. The elegance of this derivation lies in its simplicity. Another point that is interesting about this derivation is that it, along its path, makes a passing glance with the problem regarding odd perfect numbers of whether they exist or not— at equality (7), but admittedly, does not make any progress in solving it.

REFERENCE

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